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On the motion of turbulent thermals

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An analysis is presented of the motion of a turbulent thermal in an unstratified environment. Although based upon the entrainment hypothesis introduced by G. I. Taylor (see Morton, Taylor & Turner 1956), the analysis differs from previous work in that it is not limited to small density differences between the thermal and its surroundings. Also, the influence of the virtual mass of the unsteadily moving fluid, ignored by previous investigators, is included and shown to be of significance for any density difference.

Calculations of the temporal variations of size, velocity and density are presented in non-dimensional form for thermals with initial density ratios covering the practically attainable range. It is shown *a posteriori* that losses of momentum and buoyancy to a wake are probably of negligible influence in any real case.

1. Introduction

The present work was motivated by a clear inconsistency in previous models of turbulent thermals. The difficulty is most obviously revealed in the recent publication of Wang (1971), which was based on the concepts discussed in the now classical paper by Morton *et al.* (1956). In the former paper it is claimed that the analysis presented is valid for any value of the initial ratio ρ_0/ρ_∞ between the densities of the thermal (ρ_0) and its environment (ρ_∞). As this ratio approaches zero, the initial motion should be the same as for the classical case of a light sphere rising in a fluid of large density. That is, the initial acceleration of the sphere should be $2g$ (Milne-Thompson 1967), g being the gravitational acceleration, whereas Wang's analysis gives an initially infinite acceleration. Appealing to an entrainment of ambient fluid cannot explain the discrepancy. The real reason for the difficulty is quickly found by inspection of the momentum or impulse equation used by previous investigators of this problem. In all cases, the virtual mass† of the unsteadily moving thermal has been ignored.‡ As we shall show, when the influence of the virtual mass is taken into account, an internally consistent model results. We also find that it is possible to obtain an explicit analytical solution to the problem under conditions which are not unduly restrictive even in the limit of very large or small values of ρ_0/ρ_∞. The necessary

† Variously called apparent mass, added mass, virtual inertia, inertial coefficient.

‡ Although Turner (1957, 1963) has included it in analyses of some related problems, and has recently (Turner 1973) revised the model equations for the thermal problem to account for the virtual-mass effect in the limit of small density differences.

physical assumptions are discussed in some detail as are some of the implications of the model when applied to laboratory and large-scale thermals.

2. Theory

In order to present the theory as clearly as possible, the principal physical hypotheses of the model will be listed here and discussed in more detail later. These assumptions are essentially the same as those adopted by Morton *et al.* (1956).

(i) There is no loss of momentum from the thermal to a wake: this is equivalent in Wang's (1971) model to taking $C'_D = 0$.

(ii) There is no loss of buoyancy from the thermal, either to a wake or, in the case of a hot† thermal, by radiation of thermal energy to the surroundings.

(iii) Ambient fluid is entrained into the thermal at a rate proportional to its surface area and velocity. The constant of proportionality α is an entrainment coefficient with a constant value for any given thermal.

(iv) The thermal has an initially zero impulse and a spherical shape, of radius $a(t)$, and all fluid and flow properties are uniformly distributed within it.

(v) The thermal and its surroundings are composed of fluids miscible in all proportions.

(vi) Buoyancy is conserved during the entrainment process. This requirement is satisfied, for example, when the ambient fluid is of uniform density and the mixing process adiabatic and isobaric. The kinetic energy of the thermal is small compared with its total enthalpy.

Detailed discussion of the assumptions is given in §4.

It is convenient to start the analysis with the momentum equation, the correct form for which is

$$d[\tfrac{4}{3}\pi a^3(\rho + k\rho_\infty)\, u]/dt = \tfrac{4}{3}\pi a^3(\rho_\infty - \rho)\, g = \tfrac{4}{3}\pi F, \tag{1}$$

where k is the inertial coefficient for the thermal and u its velocity. The buoyancy force F is here assumed to be constant, and so

$$F \equiv a_0^3(\rho_\infty - \rho_0)\, g. \tag{2}$$

Equation (2) may be used to determine the initial radius a_0 when the initial density ρ_0 is known, e.g. for two gases with a common specific heat c_p

$$a_0^3 \equiv 3H_0/4\pi c_p T_\infty(\rho_\infty - \rho_0), \tag{3}$$

where H_0 is the total amount of thermal energy deposited in the thermal at $t = 0$ and T_∞ is the ambient temperature.

In (1), the expression in the square brackets (with $k = \frac{1}{2}$) is the correct form for the equivalent impulse of a sphere when its density differs from that of the surroundings, and reduces to the value $2\pi a^3 \rho u$ as $\rho \to \rho_\infty$ (Batchelor 1967, p. 526). With the exceptions noted in the second footnote on page 541, the inertial coefficient has been implicitly taken as zero in previous work.

† The term 'thermal' is used to connote a mass of fluid possessing buoyancy (a density difference) with respect to its surroundings, whether as a result of differences in temperature, molecular weight (for gases) or concentration of a solute (liquids).

Since F is constant, (1) can be integrated immediately to give

$$u = \left(\frac{\rho_\infty - \rho}{k\rho_\infty + \rho}\right) gt \tag{4}$$

when $u(0) = 0$. This result is the same as that which may be obtained for a rigid sphere of constant mass, although now ρ is a function of the time t as a result of entrainment, by the thermal, of fluid of density ρ_∞. It may be noted that for $\rho \ll \rho_\infty$, the case studied by Wang (1971), very large errors are introduced by setting $k = 0$, and even for the situation considered by Morton *et al.*, when $\rho \approx \rho_\infty$, the neglect of the virtual-mass term cannot be justified.

The mass conservation equation makes use of the entrainment assumption, and is

$$d(\rho a^3)/dt = 3\alpha\rho_\infty a^2|u|, \tag{5}$$

wherein we take account of the fact that the rate of entrainment is always positive, whether u is positive (upwards) or negative (downwards).

From (1) we have $\rho a^3 = \rho_\infty a^3 - F/g$, so that (5) can be simplified to give

$$da/dt = \alpha|u|, \tag{6}$$

from which, since $u = dz/dt$, we have the result

$$a - a_0 = \alpha|z|, \tag{7}$$

where z is the axial distance from its origin to the centre of the thermal at any time and $a_0 = a(0)$. Equation (7) represents the expected result that all turbulent thermals (in unstratified surroundings) have a spreading angle $2\tan^{-1}\alpha$, regardless of the initial density difference Δ_0.† Substitution for u fro m (4), and for ρ from (1), permits (6) to be integrated to give

$$\tfrac{1}{4}\rho_\infty(1+k)(a^4 - a_0^4) - (F/g)(a - a_0) = \tfrac{1}{2}\alpha|F|\,t^2. \tag{8}$$

This equation shows the important result that the spreading rate depends not only upon the magnitude $|F|$ of the buoyancy force, but also upon the sign of the density difference through the term $F/g\ (= a_0^3(\rho_\infty - \rho_0))$: it is this feature which distinguishes between the motion of heavy and light thermals.

By introducing a_0 as a characteristic length of the problem, (8) can be made non-dimensional, and a characteristic time t_c appears:

$$\tfrac{1}{4}(1+k)(\bar{a}^4 - 1) - \Delta_0(\bar{a} - 1) = \tfrac{1}{2}\bar{t}^2, \tag{9}$$

wherein $\bar{a} \equiv a/a_0$, $\Delta_0 \equiv 1 - \rho_0/\rho_\infty$ and $\bar{t} \equiv t/t_c$ with $t_c \equiv (a_0/\alpha g|\Delta_0|)^{\frac{1}{2}}$.

If we define $\bar{\Delta} \equiv (1 - \rho/\rho_\infty)/\Delta_0$ and a characteristic velocity u_c as

$$u_c \equiv a_0/\alpha t_c = (g a_0|\Delta_0|/\alpha)^{\frac{1}{2}}$$

then from (1),

$$\bar{\Delta} = 1/\bar{a}^3 \tag{10}$$

and from (4),

$$\bar{u} \equiv \frac{|u|}{u_c} = \frac{\bar{t}}{(1+k)\bar{a}^3 - \Delta_0}. \tag{11}$$

† See, however, the discussion of §4.

It is also useful to calculate the product $\overline{ua}$ since it represents the behaviour of the circulation, Reynolds number (ua/ν) and, as we shall show, the magnitude of the momentum loss to a wake (§ 4).

In the non-dimensional set of implicit equations for $\bar{a}$, $\bar{u}$, $\bar{\Delta}$ and $\bar{z}$ as functions of $\bar{t}$, only Δ_0 appears as a parameter, and a knowledge of only a_0, Δ_0 and α is required to find the behaviour of any given thermal in terms of the physical variables a, u, ρ, z and t.

Equations (9), (11) and (10) have asymptotes for short and long times as follows.

Short times: $(\bar{a}-1) \ll 1$.

$$\bar{a}-1 \approx \frac{\bar{t}^2}{2(1+k)-\Delta_0}, \quad \bar{u} \approx \frac{\bar{t}}{1+k-\Delta_0}, \quad \bar{\Delta} \approx 1.$$

Long times: $\bar{a} \gg 1$.

$$\bar{a} \approx \frac{\bar{t}^{\frac{1}{2}}}{2(1+k)^{\frac{1}{4}}}, \quad \bar{u} \approx \frac{\bar{t}^{-\frac{1}{2}}}{2^{\frac{3}{4}}(1+k)^{\frac{1}{4}}}, \quad \bar{\Delta} \approx [\tfrac{1}{2}(1+k)]^{\frac{3}{4}}\,\bar{t}^{-\frac{3}{2}}.$$

Evidently the short-time asymptotes correspond to the thermal's acceleration period, and the long-time asymptotes to its deceleration phase.

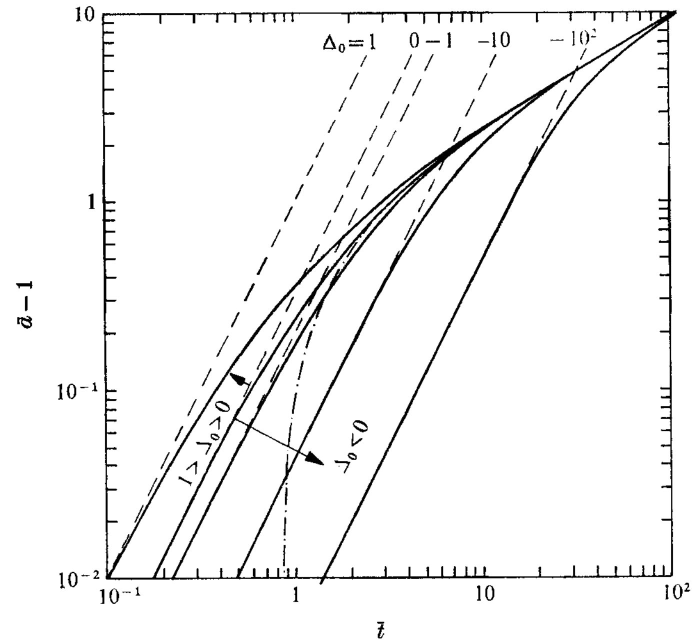
The principles used above to analyse the motion of a spherical thermal have also been applied to the case of the line thermal. An outline of this analysis is given in the appendix.

3. Results

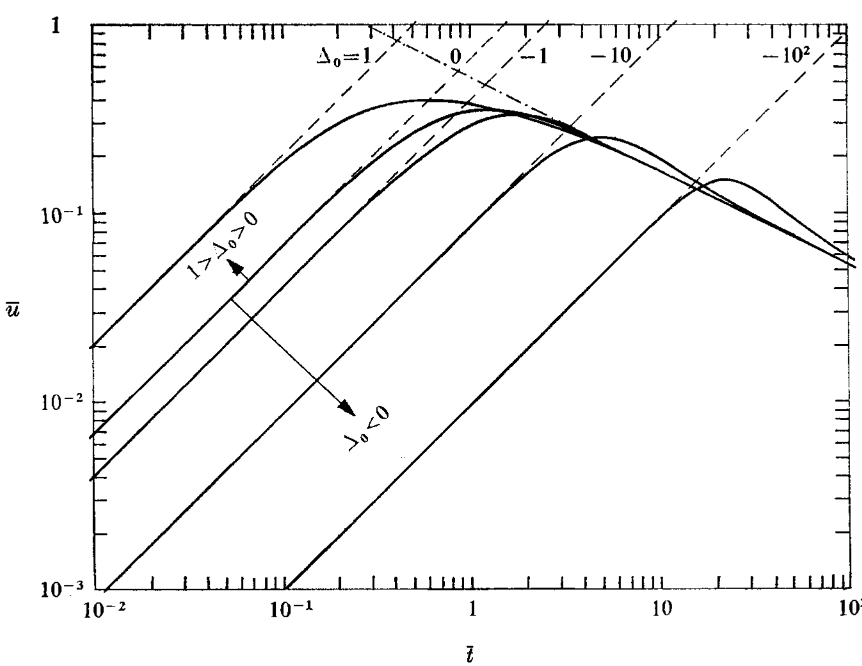
The variations in $\bar{a}$, $\bar{u}$, $\bar{\Delta}$ and $\overline{ua}$ with $\bar{t}$, as represented by (9)–(11), are plotted in figures 1 (*a*)–(*d*), together with the asymptotes for short and long times, for a range of practical values of Δ_0.† It is evident from these figures, and also the long-time asymptotes, that the motion of all thermals is eventually independent of Δ_0. Initially, however, all thermals pass through an acceleration phase which is Δ_0 dependent. The acceleration phase terminates, typically, at a value of $\bar{t}$ between 0·1 and 0·5, and there is then a transition region lasting until $\bar{t} \approx 2$ or 3 before the thermal enters its retardation phase. This sequence is essentially that described by Wang (1971), except that his numerical values for the velocity u are much larger than ours. As was pointed out earlier, for $\Delta_0 = 1$, his acceleration phase has infinite velocities, and even for $\Delta_0 = 0{\cdot}9$, which is typical of his experiments, Wang's theory gives velocities a full order of magnitude larger than those we find. Even the final asymptote for u is higher because of the neglect of the virtual mass, Wang's analysis giving $\overline{u}\bar{t}^{\frac{1}{2}} = 2^{-\frac{3}{4}}$ compared with our value of $(\frac{2}{3})^{\frac{1}{4}}\,2^{-\frac{3}{4}}$. Further comments on this and some experimental studies are postponed until § 4.

One would like to be able to decide at what stage of any experiment the thermal has reached its final asymptotic state. A statement in terms of $\bar{t}$ is difficult to transfer immediately into physical terms. An easier interpretation results from consideration of the thermal's growth. For values of $|\Delta_0| < 1$, the final stage is

† A table which includes values of Δ_0 and other parameters of interest is given in §4.

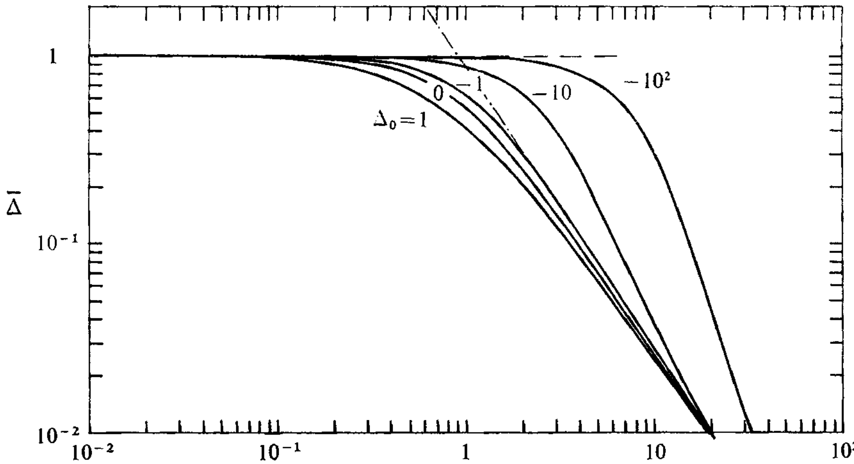


(*a*)

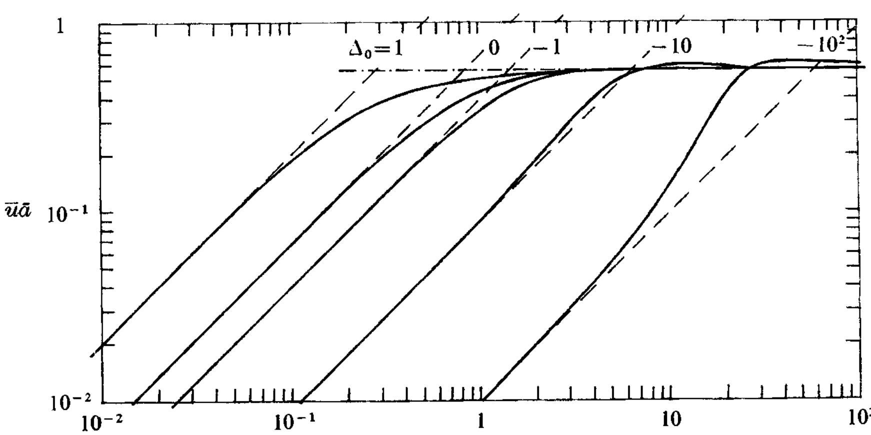


(*b*)

FIGURES 1 (*a*, *b*). For legend see next page



(c)



(d)

FIGURE 1. Variation of (a) thermal size, (b) thermal velocity, (c) thermal density and (d) thermal circulation with time. — — —, short-time asymptote; —·—·, long-time asymptote.

reached when the thermal has grown to approximately twice its original diameter. For $\Delta_0 \approx -10$, the criterion is $a/a_0 \approx 4$, for $\Delta_0 = -10^2$ the value is 6·5, and for $\Delta_0 = -10^3$ it is 15.

4. Discussion

We start the discussion by considering in order the assumptions listed in § 2 upon which we have based our analysis of the motion of a turbulent thermal.

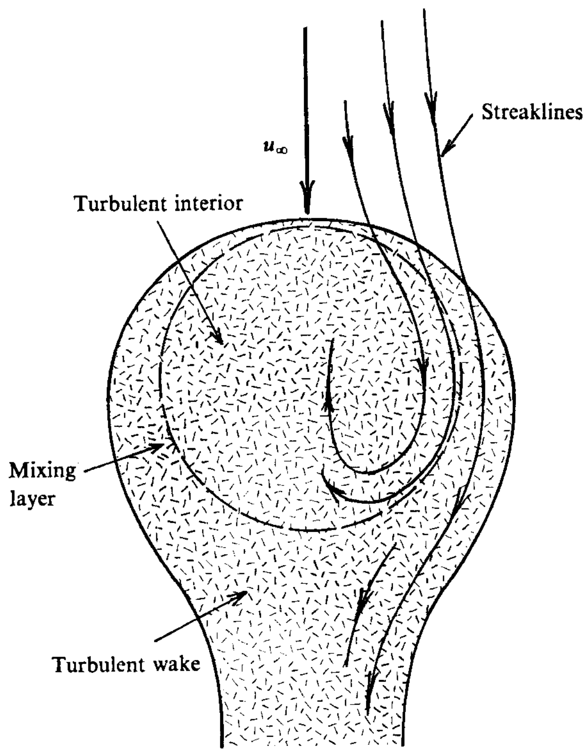


FIGURE 2. Schematic diagram to illustrate processes of entrainment and wake formation.

We can estimate the loss of impulse to a wake in much the same way as that described by Maxworthy (1972*a*) in his discussion of a laminar vortex ring. A sketch of the mixing and wake-forming process is shown in figure 2 for the present case of a turbulent thermal. By estimating the momentum loss from the mixing region as $\frac{1}{2}C_D \rho_\infty \pi a^2 u^2$,† we find that the ratio of the buoyancy force to the drag force becomes

$$8\alpha/3C_D\overline{u}^2\overline{a}^2.$$

It may be seen from figure 1(*d*) that the maximum value of $\overline{u}^2\overline{a}^2$ is close to the asymptotic value of $\frac{1}{3}$ (for $-\Delta_0 \gg 1$, $\overline{u}^2\overline{a}^2$ has a peak value slightly greater than this), and the value of α is typically $\frac{1}{4}$. Wang's (1971) estimate for C_D is certainly exaggerated: the flow around the thermal is *not* separated and the mechanisms producing vorticity in the wake are weak since they do not occur at a solid surface, but at a moving fluid interface. We estimate C_D to have a value somewhere between 10^{-1} and 10^{-2}, with no chance that it is as large as the value 0·23 associated with turbulent separation from a solid sphere (Maxworthy 1969). Even with the largest reasonable values of C_D and $\overline{u}^2\overline{a}^2$, the ratio given above exceeds 20, which we can safely say is large and so ignore the drag force in the momentum equation (1). It is easily shown, however, that the asymptotic dependence on t

† Our drag coefficient C_D is not to be confused with Wang's C_D, which also includes an 'effective drag' due to entrainment of ambient fluid, an influence taken explicitly into account in our form of the equations. We use C_D in much the same way as he uses C_D'.

(for both long and short times) is unaffected by the presence of even a large drag force. For long times the flow remains similar, with the thermal and its wake sharing the buoyancy force in constant proportions.

So far as the second assumption is concerned, for a hot thermal, there are two possible modes of buoyancy loss. If the internal temperature is very high, thermal radiation will be important during the early stages of the motion, though the data of Lin, Tsang & Wang (1972) suggest that the radiation phase will be over well before one characteristic time has elapsed. Thus one might reasonably assume that the initial enthalpy H_0 to be used in any calculation should be taken as that due to the original heating process less the amount corresponding to an almost instantaneous loss by radiation.

A much more serious difficulty is that associated with the loss of buoyancy to a wake. Available evidence (Maxworthy 1972*a*; Lin *et al.* 1972) suggests that such a loss may occur. However, experimental observations indicate that buoyancy thus rejected during the initial stages of growth in fact may catch up with the thermal as the latter rapidly slows down and, under some circumstances, may be re-entrained. The exact conditions under which this can occur are unclear at the moment. If we assume that re-entrainment does not take place, then the ratio of the buoyancy loss to the initial buoyancy can be estimated in a manner similar to that described above for the impulse, and is found to be

$$(3/\alpha)\, C_T[\bar{a}^2\bar{u}\Delta\bar{t}].\dagger$$

Values of the quantity in square brackets occurring during the initial phase are typically of order 10^{-1}, and since crude estimates from published observations show that only a small part of the initial buoyancy is lost during this period, we suggest that C_T is of order 10^{-2}. During the latter phase of the motion, the value of $\bar{a}^2\bar{u}\bar{\Delta}\bar{t}$ reaches a constant value of 0·5. Thus the total loss of buoyancy to the wake is undoubtedly small, even in the absence of re-entrainment.

A more complete analysis than that discussed here, which accounts for losses to a wake of mass, momentum and buoyancy, will be attempted at a later date.

The third assumption, relating to the entrainment coefficient α, is the most difficult to deal with since it contains all of the details of the complex turbulence processes which are such a characteristic feature of thermal motions. Until now, most investigators have assumed that α has a constant value close to 0·25. A search of the available literature, together with our own observations, suggests that α may be a function of the initial density difference parameter Δ_0,‡ and also of the detailed nature of the initial conditions under which a given thermal is formed. The spreading rate of any thermal, and so the coefficient α, seems to depend critically upon the way in which it becomes organized, for example by starting 'cleanly', or by being given an initial impulse. Examples of the latter are afforded by buoyant vortex rings, which exhibit the highest degree of

† C_T is a small coefficient (cf. a drag coefficient) which measures the effectiveness of the loss process.

‡ Dr J. S. Turner (private communication) has suggested that α might be a running variable dependent upon the *local* value of Δ. Existing experiments are not sufficiently complete to test this point. Current experiments in our laboratory give major emphasis to it, however.

organization, and the thermal produced as a result of an explosion close to the ground. The experiments of Fohl (1968) for helium releases in air ($\Delta_0 \approx 0{\cdot}85$) show that the best organized of his thermals had values of $\alpha \approx 0{\cdot}08$, while small changes in the initial conditions increased this to as much as $0{\cdot}17$. Scorer's (1957) thermals with $\Delta_0 \approx -0{\cdot}15$ gave values of α in the range $0{\cdot}2$–$0{\cdot}35$, and showed little evidence of being well organized except for a tendency to a ring-like configuration during the final stages of their travel. Maxworthy (1972*b*) found that it was possible to organize thermals in the same range of density defect as Scorer's by giving them an initial circulation greater than the final asymptotic value appropriate to their buoyancy. These thermals behaved initially like non-buoyant vortex rings but eventually approached an asymptotic state with $u \sim t^{-\frac{1}{2}}$ and a value of $\alpha \approx 0{\cdot}01$. Thermals formed in this way were stable even though the heavier fluid was located on the inside of the ring in a position one might have thought to be centrifugally unstable. By contrast, in the case of $\Delta_0 \approx 1$ (e.g. Fohl 1968), we should expect the ring to be centrifugally stable. Future experiments are planned for thermals with large negative values of Δ_0, for which we anticipate that the centrifugal instability will overcome the stabilizing effect of the core rotation and that a rapidly spreading non-organized flow will result.

The preceding conclusions, regarding the dependence of α upon initial conditions, are unfortunate for a hopefully predictive theory, since these are rarely subject to precise control, even under laboratory conditions.

The next point to be discussed was raised by Dr J. S. Turner, in private correspondence, and concerns the similarity assumption that we have made. Essentially he believes that similarity cannot exist during the initial acceleration phase of the motion since the turbulence production mechanisms have then had insufficient time to produce an entraining interface (Turner 1973, pp. 187, 196). This view is shared by Morton (1968). To us it seems that the answer depends upon the degree of abstraction one is willing to ascribe to a model. Thus, if one could produce experimentally a perfectly smooth non-turbulent fluid sphere, with zero initial impulse, then Turner's criticism would have validity. However, real experiments can never do this: turbulence is always produced both by the starting process and by the growth, due to buoyancy, of short wavelength disturbances on the thermal's surface. All of the experiments we know of show that this initial turbulent field persists through the whole range of motion. Although it may be argued that this initial turbulent field cannot be in a similar state, and there may be a time lag between the turbulence and the mean motion, to the accuracy that one can do experiments this subtlety is lost and practically the thermal acts as if it were similar with a constant value of α. In particular, we have taken measurements from Scorer's (1957) published photographs and found that the growth rate was linear through the whole range of motion, including the acceleration and transition phases.

Finally, some comments are in order regarding our assumption that the thermal is spherical in shape, and that the appropriate value for the inertial coefficient k is $\frac{1}{2}$. In many experimental situations (e.g. Fohl 1968; Wang 1971) it is observed that, after the first few diameters of its travel, the thermal takes on

Experiment	Δ_0	a_0 (m)	α	t_c (s)	u_c (m/s)	$\bar{a}_\infty$	$\bar{z}_\infty$	z_∞ (m)
Metal particles/H_2O	-10	0·01	0·25	0·02	2·0	4	16	0·16
Ba_2SO_4 and NaCl/H_2O (Scorer 1957)	$-0·15$	0·05	0·25	0·35	0·5	2	8	0·4
He/air (Fohl 1968)	~ 1	0·02	0·1	0·15	1·4	2	20	0·4
Hot air (Wang 1971)	~ 1	0·01	0·25	0·2	2·0	2	8	0·8
H.E. detonation	~ 1	10	0·1	3	30	2	20	200
(estimate)			0·25	2	20	2	8	80
Trinity explosion (Taylor 1950)	~ 1	300	0·1	15	150	2	20	6×10^3†
17 KT			0·25	11	110	2	8	$2·4 \times 10^3$

† 1 atmospheric scale height $\approx 8 \times 10^3$ m.

TABLE 1. Values of thermal parameters for laboratory and field experiments

the shape of an oblate spheroid with eccentricity $\epsilon \approx 0·1$. It is easily shown, however, that, for such a small eccentricity, both the inertial coefficient and the factors relating to the geometry of the thermal differ negligibly from the corresponding values for a sphere (Milne-Thompson 1967, p. 501). Also, it is probable that in the case of an entraining sphere the external inviscid flow may be different from that around an impermeable sphere. Thus the value of k will depend to some extent on the entrainment rate, although this again is regarded as an effect beyond the accuracy of experiments.

The results reported in §§ 2 and 3 are largely self-explanatory, and attention will be drawn to only a few points of particular importance. Table 1 shows application of the computed results to some typical experiments, both extant and planned. It includes the effect of different values of α, Δ_0, derived quantities such as t_c and u_c, and the values of a and z at which the final asymptotic stage is reached.

It is a direct consequence of the entrainment model adopted here, and by other investigators, that thermals grow linearly with z at *all* times. Thus the existence of a linear growth in an experiment, whilst it does give the value of α, is in no sense a test as to whether the final asymptotic state has been reached. Only more detailed measurements of a or u or ρ as functions of t can provide this indication. In particular, the last column of table 1 suggests that great care should be exercised in the interpretation of experimental data in terms of the asymptotic behaviour for long times, since the size of the experimental apparatus is often of the same order of magnitude as z_∞.

Our treatment provides a logical explanation for the frequently quoted statement that light thermals behave in a different way from heavy ones. Equation (4) shows explicitly that the effects of Δ_0 are not symmetrical about $\Delta_0 = 0$, and that light thermals will always have an initial acceleration larger than heavy ones (e.g. for $|\Delta_0| = 1$, the ratio is 5).

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Appendix. Line-thermal analysis

The analysis follows closely that given in § 2 for the spherical thermal. Accordingly the equations are written down here with a minimum of comment (the equation numbers have a one-to-one correspondence with those in the main body of the text).

The momentum equation is

$$d[\pi a^2(\rho+\rho_\infty)\,u]/dt = \pi a^2(\rho_\infty-\rho)\,g = \pi F, \tag{A 1}$$

where

$$F \equiv a_0^2(\rho_\infty-\rho_0)\,g \tag{A 2}$$

and

$$a_0^2 \equiv H_0/\pi c_p T_\infty(\rho_\infty-\rho_0). \tag{A 3}$$

Integrate (A 1) to find

$$u = \left(\frac{\rho_\infty-\rho}{\rho_\infty+\rho}\right) gt. \tag{A 4}$$

The mass conservation equation is

$$d(\rho a^2)/dt = 2\alpha\rho_\infty a|u|. \tag{A 5}$$

From (A 1)

$$\rho a^2 = \rho_\infty a^2 - F/g$$

and (A 5) may be simplified to give

$$da/dt = \alpha|u| = \alpha|dz/dt|, \tag{A 6}$$

from which

$$a-a_0 = \alpha|z|. \tag{A 7}$$

Substitute in (A 6) for u from (A 4), and for ρ from (A 1), and integrate to find

$$\tfrac{2}{3}\rho_\infty(a^3-a_0^3)-(F/g)\,(a-a_0) = \tfrac{1}{2}\alpha|F|\,t^2. \tag{A 8}$$

In non-dimensional form

$$\tfrac{2}{3}(\bar{a}^3-1)-\Delta_0(\bar{a}-1) = \tfrac{1}{2}\bar{t}^2. \tag{A 9}$$

The definitions of all non-dimensional variables and of the characteristic time and velocity are the same as for the spherical thermal.

From (A 1),

$$\bar{\Delta} = 1/\bar{a}^2, \tag{A 10}$$

and from (A 4),

$$\bar{u} = \bar{t}/(2\bar{a}^2-\Delta_0). \tag{A 11}$$

Short-time asymptotes: $(\bar{a}-1) \ll 1$.

$$\bar{a}-1 \approx \frac{\bar{t}^2}{2(2-\Delta_0)}, \quad \bar{u} \approx \frac{\bar{t}}{2-\Delta_0}, \quad \bar{\Delta} \approx 1.$$

Long-time asymptotes: $\bar{a} \gg 1$.

$$\bar{a} \approx (\tfrac{3}{4})^{\frac{1}{3}}\,\bar{t}^{\frac{2}{3}}, \quad \bar{u} \approx (\tfrac{2}{9})^{\frac{1}{3}}\,\bar{t}^{-\frac{1}{3}}, \quad \bar{\Delta} \approx (\tfrac{4}{3})^{\frac{2}{3}}\,\bar{t}^{-\frac{4}{3}}.$$

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